Gauge invariant formalism for second order perturbations of Schwarzschild spacetimes

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The "close limit," a method based on perturbations of Schwarzschild spacetime, has proved to be a very useful tool for finding approximate solutions to models of black hole collisions. Calculations carried out with second order perturbation theory have been shown to give the limits of applicability of the method without the need for comparison with numerical relativity results. Those second order calculations have been carried out in a fixed coordinate gauge, a method that entails conceptual and computational difficulties. Here we demonstrate a gauge invariant approach to such calculations. For a specific set of models (requiring head on collisions and quadrupole dominance of both the first and second order perturbations), we give a self contained gauge invariant formalism. Specifically, we give (i) wave equations and sources for first and second order gauge invariant wave functions; (ii) the prescription for finding Cauchy data for those equations from initial values of the first and second fundamental forms on an initial hypersurface; (iii) the formula for computing the gravitational wave power from the evolved first and second order wave functions.

I. INTRODUCTION AND OVERVIEW

In the next few years gravitational wave antennas [1–4] will go into operation with the possibility of detecting astrophysical sources. A plausible, and certainly fascinating, origin of such waves would be the powerful burst of radiation generated in the merger of two approximately equal mass black holes to form a single final hole [5]. A throrough understanding of this problem will require numerical relativity on supercomputers, and is still several years away. In the absence of numerical answers, some useful insights have already been supplied by "close limit" perturbation theory [6–12], an approximation method in which the spacetime of the merger is considered to be a perturbation of the spacetime of the single final hole. Close limit calculations, to first order in some separation parameter, have proven to give excellent agreement with numerical relativity in the case of head on collisions, models simple enough to be computed with numerical relativity. In principle one can apply the close limit method to collisions that are still beyond the scope of numerical methods, and in fact, one such result has already been reported [13].

In principle, first order perturbation theory works in the limit that some expansion parameter vanishes. In practice, the calculations are reasonably accurate for a range of that expansion parameter, up to some maximum. A shortcoming of first order perturbation theory is that there is no indication, internal to the method, of the size of that maximum. For this reason, second order close limit theory was developed [14–19]. When perturbations become large enough that the predictions of first order calculations differ significantly from the predictions of second order, it is a sign that perturbation theory is at its limit of applicability. Comparison with numerical relativity results, where they are available, has demonstrated that this method of determining "error bars" on perturbation theory is quite reliable [14–19]. Perturbation calculations must deal with the freedom to redefine coordinates, that is to do coordinate "gauge transformations." This can be approached in two very different ways. One way is to eliminate the coordinate freedom by fixing the coordinates. This, in fact, was the way in which the perturbation work on nonrotating holes [14–19] was carried out. (Except for very recent work [20], these second order calculations were done as perturbations of a nonrotating black hole.) The even parity perturbations that were the focus of these calculations, were done with a second order extension of the Zerilli [21] formalism. In the extension to second order, the Regge-Wheeler [22] conditions were used to fix the coordinates to second order as well as first. This gauge-fixed approach simplifies some of enormous complexity of the equations that arise, but a price has to be paid and paid twice. Explicit gauge transformations have to be performed to convert the initial value solution to the Regge-Wheeler gauge and an explicit transformation has to be performed to relate the computed perturbations to an asymptotically flat gauge in which information can be extracted about radiated power.

A second way of dealing with gauge freedom is to find combinations of metric perturbations that are gauge invariant, and to work only with gauge invariant quantities. The details of this procedure in first order calculations were given by Moncrief [23], who constructed a gauge invariant combination that we shall call the Moncrief first order invariant. For vacuum perturbations, this Moncrief invariant satisfies the same equation as the Zerilli wavefunction and in the Regge-Wheeler gauge, for vacuum perturbations, the Moncrief invariant reduces to the Zerilli wavefunction.

Moncrief's choice is not the unique gauge invariant that can be constructed from even parity metric perturbations,

but it is the unique choice constructed entirely from "data" on an initial hypersurface (i.e., the 3-geometry and the extrinsic curvature). This property makes it especially convenient for use in calculations of evolution of perturbations. In work of this sort we start with a solution on an initial hypersurface in some gauge (whatever gauge is convenient for solving the initial value problem). In the Moncrief formulation, the starting value of the wave function to be evolved can be evaluated in any gauge, hence it can immediately and directly be constructed from the initial value solution. A gauge invariant formulation is similarly convenient for extracting information about radiation energy carried in the perturbations. In principle this requires that one examine the perturbations in an asymptotically flat gauge. With a gauge invariant formulation, the form of the invariant can be related to an asymptotic gauge in a relatively convenient way.

The goal of the present paper is to cast the problem of second order perturbations of the Schwarzschild spacetime into a form that has the same convenience that the Moncrief formalism provides in first order perturbation calculations of evolution. That is, we will provide a second order formalism in which computations are carried out only with gauge invariant quantities, and in which these quantities are constructed from the first and second order perturbations of the spatial metric and of the extrinsic curvature for an initial hypersurface. The motivation behind this is to demonstrate the potential advantages of such a reformulation. To make this demonstration clear we choose to focus not on the general problem of perturbations of the Schwarzschild spacetime, but rather on a restricted class of problems for which a formalism can be presented with explicit details.

One restriction is that we will consider only the axisymmetric collision of nonspinning holes. Inherent in this restriction is the simplification that odd-parity perturbations vanish at every order. The decomposition of perturbations into multipoles will be used, and for simplicity we will present results only for the second order quadrupole (i.e., $\ell = 2$) perturbations. This choice is justified by the fact that most of the radiation is expected to appear in the $\ell = 2$ multipole.

A very different sort of restriction is that we shall consider only $\ell = 2$ *first* order perturbations. In principle, first order multipoles of many different orders can couple to the second order quadrupole through the nonlinear mixing of first order multipoles. One justification for ignoring the contributions of $\ell \neq 2$ first order terms is the example of close limit perturbations of collisions starting with conformally flat time symmetric initial data, such as the initial data of Brill and Lindquist [24], or especially the solution of Misner [25], the first and clearest example to which second order analysis has been applied [14–16]. For these initial data sets in the case of an axisymmetric collision of two equal mass holes, the first order perturbations are purely quadrupolar. For other initial value solutions we might expect quadrupolar first order perturbations to be larger in some sense than other first order multipoles, but this is an inadequate justification for ignoring other first order multipoles. The real justification then is to simplify the presentation of very lengthy expressions that illustrate a more generally valid approach.

In the remainder of this section we sketch out the basic ideas behind the construction of a second order gauge invariant; details will be given in the sections that follow.

We consider that we have a parameterized family of spacetime metrics of the form $g_{\alpha\beta}(x^\mu, \epsilon)$, and that these metrics can be expanded as

$$g_{\alpha\beta}(x^\mu) = g^{(0)}_{\alpha\beta}(x^\mu) + \epsilon\, g^{(1)}_{\alpha\beta}(x^\mu) + \frac{1}{2}\epsilon^2\, g^{(2)}_{\alpha\beta}(x^\mu) + \cdots\ , \tag{1}$$

where the background metric $g^{(0)}_{\alpha\beta}$ is the Schwarzschild metric in our case; $g^{(1)}_{\alpha\beta}$ is called the first order perturbation to the metric; $g^{(2)}_{\alpha\beta}$ is called the second order perturbation to the metric, and so forth. Let us now consider a parameterized family of coordinate transformations, also called transformations of the coordinate gauge, or simply "gauge transformations," $x^\mu_{new} = F^\mu(x^\alpha, \epsilon)$, that can be expanded as

$$x^\mu_{new} = x^\mu + \epsilon\xi^{(1)\mu} + \frac{1}{2}\epsilon^2\zeta^{(2)\mu} + \cdots \tag{2}$$

Such a change of coordinates will transform the metric perturbations $g^{(1)}_{\alpha\beta}, g^{(2)}_{\alpha\beta}, \cdots$. It is useful to consider special cases of the general transformation (2). If $\xi^{(1)\mu} = 0$, we call the transformation a purely second order transformation. Note that $g^{(1)}_{\alpha\beta}$ is invariant under this type of transformation, but $g^{(2)}_{\alpha\beta}, g^{(3)}_{\alpha\beta} \cdots$ are not.

The Moncrief [23] formalism is based on a certain linear combination of first order perturbations $g^{(1)}_{\alpha\beta}$ that can be determined purely from hypersurface information, i.e., from the first and second fundamental form of a hypersurface that, to zero order in ϵ is a constant time surface in the Schwarzschild geometry. Moncrief shows this combination to be gauge invariant [invariant under transformation (2)], to carry all the first order gauge invariant information, and to satisfy a simple wave equation, the Zerilli [21] equation. We use $\Psi^{(1)}$ to denote Moncrief's combination of first order perturbations $g^{(1)}_{\alpha\beta}$. [We will present this combination explicitly below in (19) after we have introduced multipole

decomposition and the appropriate notation.] We will use $L^{(2)}$ to represent the same combination of second order perturbations $g^{(2)}_{\alpha\beta}$. It follows immediately that the second order combination $L^{(2)}$ is invariant under purely second order gauge transformations. Since $\Psi^{(1)}$ is constructed from the first order perturbations of the spatial geometry and extrinsic curvature of a hypersurface, it follows that $L^{(2)}$ can also be constructed from hypersurface information. We next turn to the question of the wave equation satisfied by $L^{(2)}$.

The vacuum Einstein equations can be written as

$$\widehat{E}(g_{\alpha\beta}) = \widehat{E}(g^{(0)}_{\alpha\beta} + \epsilon\, g^{(1)}_{\alpha\beta} + \frac{1}{2}\epsilon^2\, g^{(2)}_{\alpha\beta} + \cdots) = 0 \tag{3}$$

where $\widehat{E}$ represents the set of nonlinear differential operators that generates the Einstein equations. The terms in (3) that are zero order in ϵ will be satisfied automatically because $g^{(0)}_{\alpha\beta}$, the background metric, is a solution to the vacuum Einstein equations. To find the equations satisfied by the first order perturbation, we expand (3) in powers of ϵ, and write the set of first order equations as

$$\widehat{O}(g^{(1)}_{\alpha\beta}) = 0\ . \tag{4}$$

Since the perturbations $g^{(1)}_{\alpha\beta}$ can only appear linearly, $\widehat{O}$ represents a set of linear differential operators.

The second order part of the expansion of (3) will involve terms linear in second order perturbations and terms involving products of first order perturbations. These equations can be written symbolically as

$$\widehat{O}(g^{(2)}_{\alpha\beta}) = \widehat{S}(g^{(1)}_{\alpha\beta}, g^{(1)}_{\alpha\beta})\ . \tag{5}$$

In this form, the products of first order terms appear on the right. If the perturbative problem is solved order by order, the first order problem may be considered already to have been solved, so that the right hand side of (5) can be considered as known. It should especially be noticed that the operator $\widehat{O}$ is "zero" order. That is, $\widehat{O}$ is precisely the same operator that appears in the first order equations (4). One can view (5) as a system of differential equations differing from (4) only by the presence of known source terms. We know that the first order equations can be rearranged into a single wave equation, the Zerilli equation, which we symbolize as

$$\widehat{Z}(\Psi^{(1)}) = 0\ . \tag{6}$$

It follows that the equations of (5) can be rearranged to give a single wave equation

$$\widehat{Z}(L^{(2)}) = \widehat{S}_{\rm Mon}(g^{(1)}_{\alpha\beta}, g^{(1)}_{\alpha\beta})\ , \tag{7}$$

in which the differential operator $\widehat{Z}$ is the Zerilli operator. The right hand side represents the set of terms quadratic in first order perturbations, that result from forming the Zerilli equation for the Moncrief combination. These first order terms can be viewed as known once the first order perturbation problem has been solved, so (7) is to be viewed as a wave equation for $L^{(2)}$ with a known source.

Though the quantity $L^{(2)}$ is constructed from hypersurface information and satisfies a convenient equation, it is not what we seek. We have seen that it is invariant under purely second order perturbations, but it is not invariant under more general gauge transformations. That is, $L^{(2)}$ will in general change under a transformation (2) with $\xi^{(1)} \neq 0$. In order to construct a second order perturbation function that *is* gauge invariant we must add another type of expression to $L^{(2)}$. Let $Q^{(1)}$ represent any combination of products of first order perturbations. Since the operator $\widehat{Z}$ is linear, for any such $Q^{(1)}$ the quantity

$$\Psi^{(2)} \equiv L^{(2)} + Q^{(1)} \tag{8}$$

will satisfy an equation of the form

$$\widehat{Z}(\Psi^{(2)}) = \widehat{Z}(L^{(2)} + Q^{(1)}) = \widehat{S}_{\rm Mon}(g^{(1)}_{\alpha\beta}, g^{(1)}_{\alpha\beta}) + \widehat{Z}(Q^{(1)})\ . \tag{9}$$

The added source term $\widehat{Z}(Q^{(1)})$ is known once the first order problem is solved, so $\Psi^{(2)}$, like $L^{(2)}$, satisfies a Zerilli equation with a known source. One of the main points of this paper is to display explicit forms of $Q^{(1)}$ for which $\Psi^{(2)}$ is gauge invariant for the general gauge transformation in (2). It should be noted at the outset that $Q^{(1)}$ cannot be unique, and hence $\Psi^{(2)}$ cannot be unique. To see this, consider $\Xi(\Psi^{(1)}, \Psi^{(1)})$ to be any quadratic combination of terms in $\Psi^{(1)}$, then define

$$\Psi^{(2)}_{\rm alt} = \Psi^{(2)} + \Xi(\Psi^{(1)}, \Psi^{(1)}) \ , \tag{10}$$

and note the following: (i) $\Psi^{(1)}$ is gauge invariant under first and second order transformations, hence $\Psi^{(2)}_{\rm alt}$ is first and second order gauge invariant if $\Psi^{(2)}$ is first and second order gauge invariant. (ii) Since Moncrief's $\Psi^{(1)}$ is constructed completely from hypersurface data, it follows that this is true also of $\Psi^{(2)}_{\rm alt}$ if it is true of $\Psi^{(2)}$. (iii) Like the original invariant, the alternate invariant $\Psi^{(2)}_{\rm alt}$ satisfies a Zerilli wave equation

$$\widehat{Z}(\Psi^{(2)}_{\rm alt}) = \widehat{Z}\left(L^{(2)} + Q^{(1)} + \Xi(\Psi^{(1)}, \Psi^{(1)})\right) = \widehat{S}_{\rm Mon}(g^{(1)}_{\alpha\beta}, g^{(1)}_{\alpha\beta}) + \widehat{Z}(Q^{(1)}) + \widehat{Z}\left(\Xi(\Psi^{(1)}, \Psi^{(1)})\right) \ , \tag{11}$$

with a known source.

The remainder of this paper will be organized as follows. In Sec. II the perturbed metric tensor and wave equations, to first and second order, are introduced. The details of first and second order gauge transformations are discussed in Sec. III and a wave function is presented that is invariant under these transformations. The procedure for finding Cauchy data for this wave function is given in Sec. IV. The relationship of the invariant wavefunction to gravitational wave energy is analyzed in Sec. V, and a summary and discussion are given in Sec. VI. Throughout the paper we use the conventions of Misner *et al.* [26]. In particular we use a metric with sign conventions -+++, and units in which $c = G = 1$.

II. PERTURBED METRIC TENSOR AND MONCRIEF WAVE EQUATION FOR SECOND ORDER

A. Perturbed metric tensor

As was discussed in Sec. I, to both first and second order we consider only $\ell = 2$ multipoles. Using the standard Regge-Wheeler [22] notation we can write the perturbations [the $g^{(1)}_{\alpha\beta}$of (1)] to the Schwarzschild metric for mass M, as

$$g_{tt} = -(1-2M/r)\left[1 - \left\{\epsilon H_0^{(1)}(r,t) + \frac{\epsilon^2}{2}H_0^{(2)}(r,t)\right\} P_2(\theta)\right] \tag{12}$$

$$g_{tr} = \left[\epsilon H_1^{(1)}(r,t) + \frac{\epsilon^2}{2}H_1^{(2)}(r,t)\right] P_2(\theta) \tag{13}$$

$$g_{t\theta} = \left[\epsilon h_0^{(1)}(r,t) + \frac{\epsilon^2}{2}h_0^{(2)}(r,t)\right] P_2^{'}(\theta) \tag{14}$$

$$g_{rr} = (1-2M/r)^{-1}\left[1 + \left\{\epsilon H_2^{(1)}(r,t) + \frac{\epsilon^2}{2}H_2^{(2)}(r,t)\right\} P_2(\theta)\right] \tag{15}$$

$$g_{r\theta} = \left[\epsilon h_1^{(1)}(r,t) + \frac{\epsilon^2}{2}h_1^{(2)}(r,t)\right] P_2^{'}(\theta) \tag{16}$$

$$g_{\theta\theta} = r^2\left[1 + \left\{\epsilon K^{(1)}(r,t) + \frac{\epsilon^2}{2}K^{(2)}(r,t)\right\} P_2(\theta) + \left\{\epsilon G^{(1)}(r,t) + \frac{\epsilon^2}{2}G^{(2)}(r,t)\right\} P_2^{''}(\theta)\right] \tag{17}$$

$$g_{\phi\phi} = r^2\left[\sin^2\theta + \left\{\epsilon K^{(1)}(r,t) + \frac{\epsilon^2}{2}K^{(2)}(r,t)\right\}\sin^2\theta P_2(\theta) + \left\{\epsilon G^{(1)}(r,t) + \frac{\epsilon^2}{2}G^{(2)}(r,t)\right\}\sin\theta\cos\theta P_2^{'}(\theta)\right] . \tag{18}$$

Here we are using $P_2(\theta)$ to denote the Legendre polynomial of order 2, with argument $\cos\theta$. By $P_2^{'}(\theta)$ and $P_2^{''}(\theta)$ we mean respectively the first and second derivative of $P_2(\theta)$ with respect to θ. We include an upper index between parentheses whenever it is necessary to clarify the order of a perturbation quantity.

B. Zerilli wave equation

As described in Sec. I, a starting point in the search for second order invariants is the first order Moncrief invariant $\Psi^{(1)}$. For $\ell = 2$, in terms of the Regge-Wheeler notation introduced above, $\Psi^{(1)}$ is the following linear combination of first order perturbations

$$\Psi^{(1)} \equiv \frac{r}{6\,(2r+3M)}\left[2(r-2M)(H_2^{(1)}-r\partial_r K^{(1)})-2(r-3M)K^{(1)}+6\left\{rK^{(1)}+\frac{(r-2M)}{r}(r^2\partial_r G^{(1)}-2h_1^{(1)})\right\}\right]\ . \quad (19)$$

Moncrief has shown that this first order combination is invariant under gauge transformation (2) and satisfies the Zerilli [21] equation

$$\widehat{Z}(\Psi^{(1)}) = 0\ . \quad (20)$$

Here the Zerilli operator $\widehat{Z}$ is

$$\widehat{Z} = -\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r_*^2}+V(r)\ , \quad (21)$$

where r_* is the usual "tortoise" coordinate covering the exterior of the black hole,

$$r_* = r + 2M\ln(r/2M-1)\ , \quad (22)$$

so that the horizon is at $r_* = -\infty$ and spatial infinity is at $r_* = \infty$. The potential term in the $\ell = 2$ Zerilli operator is given by

$$V(r) = 6\left(1-\frac{2M}{r}\right)\frac{4r^3+4r^2M+6rM^2+3M^3}{r^3(2r+3M)^2}\ . \quad (23)$$

We now define $L^{(2)}$ to be the second order equivalent of $L^{(1)}$:

$$L^{(2)} \equiv \frac{r}{6\,(2r+3M)}\left[2(r-2M)(H_2^{(2)}-r\partial_r K^{(2)})-2(r-3M)K^{(2)}+6\left\{rK^{(2)}+\frac{(r-2M)}{r}(r^2\partial_r G^{(2)}-2h_1^{(2)})\right\}\right]\ . \quad (24)$$

As was argued in connection with (7), this second order combination satisfies an equation of the form (7) with $\widehat{S}_{\rm Mon}$ a sum of products of the first order perturbations, $H_0^{(1)}$, $H_1^{(1)}$, $H_2^{(1)}$, $h_0^{(1)}$, $h_1^{(1)}$, $K^{(1)}$, $G^{(1)}$, and the derivatives of these functions. The explicit form of $\widehat{S}_{\rm Mon}$ is straightforward to compute; one simply repeats the steps that lead to the first order Zerilli equation (20) and keeps all terms of second order. But the result is extremely lengthy and will not be displayed here.

It is worth noting that $\widehat{S}_{\rm Mon}$ is automatically invariant for purely second order transformations since first order perturbations do not change for purely second order transformations. Since $\widehat{Z}$ is invariant under general gauge transformations, (7) then tells us that $L^{(2)}$ must be gauge invariant under purely second order transformations. This is a property that also follows from the manner in which $L^{(2)}$ is constructed. The validity of (7) can then be viewed as a check of consistency.

III. SECOND ORDER INVARIANT WAVEFUNCTION

To explore the gauge changes in metric perturbations we must introduce a specific form of a gauge transformation. The form of the transformation in (2) was used in the studies by Gleiser *et al.* [14–16]. Here we choose instead the equivalent form of higher order gauge transformation given by Bruni *et al.* [27]

$$x^\alpha_{new} = x^\alpha + \epsilon\,\xi^{(1)\alpha}(x^\beta) + (\epsilon^2/2)\ \left[\xi^{(2)\alpha}(x^\beta) + \partial_\mu\xi^{(1)\alpha}(x^\beta)\ \xi^{(1)\mu}(x^\beta)\right]\ . \quad (25)$$

The reason for this choice is a practical one. With this notation $\xi^{(1)\alpha}$ and $\xi^{(2)\alpha}$ can be treated as generating vectors and the first and second order gauge transformations can be written in the form of Lie derivatives

$$\delta g^{(1)}_{\alpha\beta} = \mathcal{L}_{\xi^{(1)}}\, g^{(0)}_{\alpha\beta} \quad (26)$$

$$\delta g^{(2)}_{\alpha\beta} = (\mathcal{L}_{\xi^{(2)}} + \mathcal{L}^2_{\xi^{(1)}})\, g^{(0)}_{\alpha\beta} + 2\mathcal{L}_{\xi^{(1)}}\, g^{(1)}_{\alpha\beta}\ . \quad (27)$$

Lie derivatives of tensors can be handled automatically by the Maple symbolic manipulation language that was used to do the computations.

Since we are using multipole decomposition and keeping only the quadrupole terms, we write the components of the generating vectors as

$$\xi^{(1)} = \left\{C_t^{(1)}(r,t)\, P_2(\theta),\ C_r^{(1)}(r,t)\, P_2(\theta),\ C_\theta^{(1)}(r,t)\, P_2^{'}(\theta),\ 0\right\} \tag{28}$$

$$\xi^{(2)} = \left\{C_t^{(2)}(r,t)\, P_2(\theta),\ C_r^{(2)}(r,t)\, P_2(\theta),\ C_\theta^{(2)}(r,t)\, P_2^{'}(\theta),\ 0\right\}\ . \tag{29}$$

The first order gauge transformations then take the form

$$\delta H_2^{(1)} = 2\partial_r C_r^{(1)}(r,t) - r^{-2}\,(1-2M/r)^{-1}\, 2M C_r^{(1)}(r,t) \tag{30}$$

$$\delta h_1^{(1)} = (1-2M/r)^{-1}\, C_r^{(1)}(r,t) + r^2\partial_r C_\theta^{(1)}(r,t) \tag{31}$$

$$\delta K^{(1)} = 2r^{-1} C_r^{(1)}(r,t) \tag{32}$$

$$\delta G^{(1)} = 2C_\theta^{(1)}(r,t) \tag{33}$$

$$\delta H_1^{(1)} = (1-2M/r)^{-1}\,\partial_t C_r^{(1)}(r,t) - (1-2M/r)\,\partial_r C_t^{(1)}(r,t) \tag{34}$$

$$\delta h_0^{(1)} = -\,(1-2M/r)\, C_t^{(1)}(r,t) + r^2\partial_t C_\theta^{(1)}(r,t)\ . \tag{35}$$

Pure second order gauge transformations would look exactly like (30)–(35), replacing upper index 1 by 2. An example of general second order gauge transformations, after projecting into $\ell = 2$, is

$$\begin{aligned}\delta G^{(2)} = {}& 2C_\theta^{(2)} + \frac{(-2)}{7r^3(r-2M)}(-r^3(r-2M)C_t^{(1)}\partial_t C_\theta^{(1)} - \\ & 3r^2 C_r^{(1)} C_r^{(1)} - 18r^3(r-2M)C_\theta^{(1)}C_\theta^{(1)} + 2r^3(r-2M)C_r^{(1)}\partial_r G^{(1)} - \\ & 6r(r-2M)h_1^{(1)}C_r^{(1)} - 6r(r-2M)h_0^{(1)}C_t^{(1)} + 3(r-2M)^2 C_t^{(1)}C_t^{(1)} + 2r^3(r-2M)C_t^{(1)}\partial_t G^{(1)} + \\ & 4r^3(r-2M)C_\theta^{(1)}K^{(1)} - 18r^3(r-2M)C_\theta^{(1)}G^{(1)} + \\ & 4r^2(r-2M)C_r^{(1)}G^{(1)} - r^3(r-2M)C_r^{(1)}\partial_r C_\theta^{(1)} + 8r^2(r-2M)C_\theta^{(1)}C_r^{(1)})\ .\end{aligned} \tag{36}$$

Expressions like (36) generate the first order gauge transformations of the linear part in second order perturbations of $L^{(2)}$.

The key to building invariants at second order is to take combinations of the gauge transformation equations in (30)–(35) that isolate the coefficient functions occurring in (28) and (29),

$$\delta[h_1^{(1)} - \frac{r^2}{2}\partial_r G^{(1)}] = \frac{r}{(r-2M)}C_r^{(1)} \tag{37}$$

$$\delta[h_0^{(1)} - \frac{r^2}{2}\partial_t G^{(1)}] = -\frac{(r-2M)}{r}C_t^{(1)} \tag{38}$$

$$\delta[G^{(1)}] = 2C_\theta^{(1)}\ . \tag{39}$$

With these at hand one can construct first order quadratic terms that, under a first order transformation, cancel gauge dependent terms arising from the transformation of $L^{(2)}$. This procedure leads to the following as the simplest choice for a first and second order gauge invariant.

$$\Psi_{\rm RW}^{(2)} \equiv L^{(2)} + Q_{\rm RW}^{(1)} \tag{40}$$

where $L^{(2)}$ is the second order equivalent of $\Psi^{(1)}$, given in (24), and

$$\begin{aligned}Q_{\rm RW}^{(1)} \equiv {}& \frac{4}{21r^3(r-2M)(2r+3M)}\Big[(r-2M)^2(h_1^{(1)} - \frac{r^2}{2}\partial_r G^{(1)})(-2r(3r+M)K^{(1)} + 6r(r-2M)\partial_r h_1^{(1)} + \\ & 6r(5r+4M)G^{(1)} - 7r^2 M\partial_r K^{(1)} + r(3r+2M)H_2^{(1)} - 3r^3(r-2M)\partial_{r^2}G^{(1)} - 6(5r+4M)h_1^{(1)} + \\ & 6r^2(r+5M)\partial_r G^{(1)} + r^3(r-2M)\partial_{r^2}K^{(1)} - r^2(r-2M)\partial_r H_2^{(1)}) + \\ & r^2(r-2M)^2\partial_r(\frac{(r-2M)}{r}(h_1^{(1)} - \frac{r^2}{2}\partial_r G^{(1)}))(-2rH_2^{(1)} + 6h_1^{(1)} - 3r^2\partial_r G^{(1)} - 6rG^{(1)} + r^2\partial_r K^{(1)} + 2rK^{(1)}) + \\ & \frac{r^2(r-2M)}{2}G^{(1)}(18(r-2M)h_1^{(1)} - 3r^2(r-2M)\partial_r G^{(1)} - 3r^2(r-2M)\partial_r K^{(1)} - 24r(2r+3M)G^{(1)} +\end{aligned}$$

$$
\begin{aligned}
&9r(2r+3M)K^{(1)} - 3r(r-2M)H_2^{(1)}) + \frac{6r^2(r-2M)^2}{2}(r^2G^{(1)} - (r-2M)h_1^{(1)})\partial_r G^{(1)} + \\
&r(r-2M)(\frac{(-r)}{(r-2M)}(h_0^{(1)} - \frac{r^2}{2}\partial_t G^{(1)}))(-r^2(r-2M)\partial_t H_2^{(1)} + r^3(r-2M)\partial_{rt}K^{(1)} - r^2(2r+3M)\partial_t K^{(1)} - \\
&3r^3(r-2M)\partial_{rt}G^{(1)} + 3r(r-2M)\partial_t h_1^{(1)} + 3r(r-2M)H_1^{(1)} + 3r(r-2M)\partial_r h_0^{(1)} + 6r^2(2r+3M)\partial_t G^{(1)} - \\
&6(5r+4M)h_0^{(1)}) + \\
&r^2(r-2M)^2\partial_r(\frac{(-r)}{(r-2M)}(h_0^{(1)} - \frac{r^2}{2}\partial_t G^{(1)}))(-3r^2\partial_t G^{(1)} + 6h_0^{(1)} - 2(r-2M)H_1^{(1)} + r^2\partial_t K^{(1)}) + \\
&\frac{(r-2M)^2}{(2r)}(38r^2 + 28rM + 4M^2)(h_1^{(1)} - \frac{r^2}{2}\partial_r G^{(1)})(h_1^{(1)} - \frac{r^2}{2}\partial_r G^{(1)}) - \\
&8(r-2M)^2r^2(h_1^{(1)} - \frac{r^2}{2}\partial_r G^{(1)})\partial_r(\frac{(r-2M)}{r}(h_1^{(1)} - \frac{r^2}{2}\partial_r G^{(1)})) - \\
&6r(r-2M)^2(5r+4M)(h_1^{(1)} - \frac{r^2}{2}\partial_r G^{(1)})G^{(1)} + \\
&3r^2(r-2M)^3(h_1^{(1)} - \frac{r^2}{2}\partial_r G^{(1)})\partial_r G^{(1)} + \\
&r^3(r-2M)^2\partial_r(\frac{(r-2M)}{r}(h_1^{(1)} - \frac{r^2}{2}\partial_r G^{(1)}))\partial_r(\frac{(r-2M)}{r}(h_1^{(1)} - \frac{r^2}{2}\partial_r G^{(1)})) + \\
&6r^3(r-2M)^2\partial_r(\frac{(r-2M)}{r}(h_1^{(1)} - \frac{r^2}{2}\partial_r G^{(1)}))G^{(1)} + \\
&6r^3(2r+3M)(r-2M)G^{(1)}G^{(1)} - 3r^4(r-2M)^2\partial_r G^{(1)}G^{(1)} + \frac{3}{4}r^4(r-2M)^3\partial_r G^{(1)}\partial_r G^{(1)} - \\
&3(5r+3M)(r-2M)^2(\frac{(-r)}{(r-2M)}(h_0^{(1)} - \frac{r^2}{2}\partial_t G^{(1)}))(\frac{(-r)}{(r-2M)}(h_0^{(1)} - \frac{r^2}{2}\partial_t G^{(1)})) + \\
&6r(r-2M)^3(\frac{(-r)}{(r-2M)}(h_0^{(1)} - \frac{r^2}{2}\partial_t G^{(1)}))\partial_r(\frac{(-r)}{(r-2M)}(h_0^{(1)} - \frac{r^2}{2}\partial_t G^{(1)})) - \\
&r(r-2M)^4\partial_r(\frac{-r}{(r-2M)}(h_0^{(1)} - \frac{r^2}{2}\partial_t G^{(1)}))\partial_r(\frac{-r}{(r-2M)}(h_0^{(1)} - \frac{r^2}{2}\partial_t G^{(1)}))\Big] \ . \qquad (41)
\end{aligned}
$$

In the Regge-Wheeler choice of gauge [22] the perturbation functions $h_0^{(1)}$, $h_1^{(1)}$ and $G^{(1)}$ are chosen to vanish. In this gauge the right hand side of (41) vanishes. For this reason we denote by $Q_{\rm RW}^{(1)}$ the particular choice of quadratic terms appearing in (41) and we denote the corresponding wave function as $\Psi_{\rm RW}^{(2)}$. Note that $\Psi_{\rm RW}^{(2)}$ becomes simply $L^{(2)}$ in the Regge-Wheeler gauge. The wave function $\Psi_{\rm RW}^{(2)}$ satisfies a wave equation

$$
\widehat{Z}(\Psi_{\rm RW}^{(2)}) = \widehat{Z}(L^{(2)} + Q_{\rm RW}^{(1)}) = \widehat{S}_{\rm Mon}(g_{\alpha\beta}^{(1)}, g_{\alpha\beta}^{(1)}) + \widehat{Z}(Q_{\rm RW}^{(1)}) \equiv \mathcal{S}_{\rm RW} \ . \qquad (42)
$$

If one works explicitly in the Regge-Wheeler gauge, then the terms in $\widehat{Z}(Q_{\rm RW}^{(1)})$ vanish.

The explicit expression for the source term is extremely lengthy. To compress it into a manageable form we introduce a few simplifications in notation. We use a prime ($'$) to denote partial differentiation with respect to r and a dot ($\dot{}$) to denote partial differentiation with respect to time, and we write it in terms of two first order quantities K and H_2:

$$
\begin{aligned}
S_{\rm RW} = (-2/189)\Big[&(1512M^5 - 4068r^3M^2 + 1602r^4M - 252r^5 - 4248rM^4 + 5490r^2M^3)H_2K' + \\
&(234r^4 - 972r^3M + 324M^4 - 468rM^3 + 1161r^2M^2)KK + \\
&(30r^6 - 189r^2M^4 + 147r^3M^3 - 77r^4M^2 + 432rM^5 - 52r^5M)\dot{K}\dot{K} + \\
&(-9r^6 + 72r^5M - 216r^4M^2 + 288r^3M^3 - 144r^2M^4)K'K' + \\
&(-36r^6 - 864r^4M^2 + 288r^5M + 1152r^3M^3 - 576r^2M^4)KK'' + \\
&(729r^4M^2 - 459r^5M + 72r^6 + 882r^3M^3 + 2376rM^5 - 3348r^2M^4)H_2'K' + \\
&(-783r^4M^2 + 90r^5M + 2538r^3M^3 + 1944rM^5 - 3636r^2M^4)H_2K'' + \\
&(-306r^4M + 1944r^3M^2 - 864M^5 + 3744rM^4 - 4320r^2M^3)H_2H_2' +
\end{aligned}
$$

$$(1575r^2M^2 + 387r^4 - 1359Mr^3 + 3132M^4 - 2376rM^3)H_2H_2 +$$
$$(-450r^3M + 1080M^4 + 2070r^2M^2 - 2880rM^3)H_2K +$$
$$(-36r^6 - 87r^4M^2 + 48r^5M + 540r^2M^4)H_2\ddot{K} +$$
$$(36r^6 - 276r^4M^2 + 162r^3M^3 + 540r^2M^4 - 42r^5M)K\ddot{K} +$$
$$(-117r^6M + 18r^7 - 576r^3M^4 + 216r^5M^2 + 72r^4M^3 + 432r^2M^5)K^{'}H_2^{''} +$$
$$(-117r^6M + 18r^7 + 72r^4M^3 - 576r^3M^4 + 432r^2M^5 + 216r^5M^2)H_2K^{'''} +$$
$$(-56r^6M^3 - 2r^9 + 48r^7M^2 - 3r^8M - 96r^5M^4 + 144r^4M^5)\dot{K}\dot{K}''' +$$
$$(-54r^6M + 12r^7 + 120r^4M^3 - 144r^3M^4 + 36r^5M^2)K\ddot{H}_2' +$$
$$(-36r^6M^2 + 54r^7M - 12r^8 + 144r^4M^4 - 120r^5M^3)K\ddot{K}'' +$$
$$(-60r^6M^3 - 18r^7M^2 - 6r^9 + 27r^8M + 72r^5M^4)K^{'}\ddot{K}'' +$$
$$(-18r^7M^2 - 6r^9 + 27r^8M + 72r^5M^4 - 60r^6M^3)K^{''}\ddot{K}' +$$
$$(-48r^7M^2 - 16r^6M^3 + 26r^8M - 4r^9 - 96r^4M^5 + 128r^5M^4)\dot{K}'\dot{H}_2'' +$$
$$(16r^7M^3 + 4r^{10} - 128r^6M^4 - 26r^9M + 48r^8M^2 + 96r^5M^5)\dot{K}'\dot{K}''' +$$
$$(3r^7M - 48r^6M^2 + 2r^8 + 96r^4M^4 - 144r^3M^5 + 56r^5M^3)\dot{K}\dot{H}_2'' +$$
$$(151r^7M - 258r^6M^2 - 22r^8 + 1288r^4M^4 - 960r^3M^5 - 316r^5M^3)\dot{K}'\dot{H}_2' +$$
$$(18r^7 + 588r^3M^4 - 720r^2M^5 - 133r^5M^2 + 124r^4M^3 - 29r^6M)\dot{H}_2'\dot{K} +$$
$$(-108r^5 + 792r^4M - 2214r^3M^2 + 432M^5 - 1800rM^4 + 2916r^2M^3)KK^{'} +$$
$$(171r^6M - 24r^7 + 2376r^2M^5 - 1362r^3M^4 - 211r^4M^3 - 97r^5M^2)\dot{K}'\dot{K} +$$
$$(57r^6M - 6r^7 + 378r^3M^4 - 99r^4M^3 - 135r^5M^2)K^{'}\ddot{K} +$$
$$(432M^5 - 36r^4M + 162r^3M^2 - 360rM^4 - 108r^2M^3)H_2^{'}K +$$
$$(-18r^6 - 72r^3M^3 + 117r^5M - 216r^4M^2 - 432rM^5 + 576r^2M^4)H_2H_2^{''} +$$
$$(36r^6 + 144r^3M^3 - 234r^5M + 432r^4M^2 - 1152r^2M^4 + 864rM^5)KH_2^{''} +$$
$$(-128r^6M^4 + 4r^{10} + 16r^7M^3 + 48r^8M^2 - 26r^9M + 96r^5M^5)\dot{K}''\dot{K}'' +$$
$$(160r^6M^2 - 17r^8 - 6r^7M + 1992r^3M^5 - 2264r^4M^4 + 474r^5M^3)\dot{K}'\dot{K}' +$$
$$(4r^8 - 26r^7M + 96r^3M^5 - 128r^4M^4 + 48r^6M^2 + 16r^5M^3)\dot{H}_2'\dot{H}_2' +$$
$$(48r^6M^2 - 26r^7M + 4r^8 + 96r^3M^5 - 128r^4M^4 + 16r^5M^3)\dot{H}_2\dot{H}_2'' +$$
$$(-16r^6M^3 - 48r^7M^2 - 4r^9 + 26r^8M - 96r^4M^5 + 128r^5M^4)\dot{H}_2\dot{K}''' +$$
$$(-18r^6M^2 + 27r^7M - 6r^8 + 72r^4M^4 - 60r^5M^3)H_2^{'}\ddot{K}' +$$
$$(-54r^6M + 12r^7 - 144r^3M^4 + 120r^4M^3 + 36r^5M^2)H_2\ddot{H}_2' +$$
$$(-57r^6M + 6r^7 - 252r^3M^4 + 120r^4M^3 + 93r^5M^2)K^{'}\ddot{H}_2 +$$
$$(-36r^6M^2 + 144r^4M^4)H_2\ddot{K}'' +$$
$$(121r^7M - 18r^8 - 816r^3M^5 + 1120r^4M^4 - 192r^6M^2 - 312r^5M^3)\dot{H}_2\dot{K}'' +$$
$$(33r^7M - 6r^8 + 108r^4M^4 - 72r^5M^3 - 33r^6M^2)K^{''}\ddot{K} +$$
$$(180r^6M - 24r^7 - 384r^4M^3 + 720r^3M^4 - 252r^5M^2)K\ddot{K}' +$$
$$(-27r^6M + 6r^7 - 72r^3M^4 + 60r^4M^3 + 18r^5M^2)H_2^{'}\ddot{H}_2 +$$
$$(-144r^6M^2 + 117r^7M - 18r^8 - 252r^5M^3 + 432r^4M^4)K^{'}\ddot{K}' +$$
$$(-32r^6M^3 - 96r^7M^2 + 52r^8M - 8r^9 - 192r^4M^5 + 256r^5M^4)\dot{H}_2'\dot{K}'' +$$
$$(232r^6M^3 + 384r^7M^2 - 179r^8M + 22r^9 + 1296r^4M^5 - 1568r^5M^4)\dot{K}'\dot{K}'' +$$
$$(182r^6M^2 + 43r^7M - 18r^8 + 1224r^3M^5 - 672r^4M^4 - 362r^5M^3)\dot{K}\dot{K}'' +$$
$$(54r^7M - 12r^8 - 120r^5M^3)H_2\ddot{K}'' +$$

$$
\begin{aligned}
&(-27r^7M+6r^8-72r^4M^4+60r^5M^3+18r^6M^2)K^{''}\ddot{H}_2+\\
&(60r^7+2604r^3M^4-1584r^2M^5+67r^5M^2-988r^4M^3-133r^6M)\dot{H}_2\dot{K}'+\\
&(12r^7+108r^3M^4+36r^4M^3-69r^5M^2-12r^6M)H_2^{'}\ddot{K}+\\
&(-42r^7+720r^3M^4-492r^4M^3-216r^5M^2+225r^6M)H_2\ddot{K}'+\\
&(-24r^6-78r^3M^3-120r^5M+168r^4M^2-576rM^5+1116r^2M^4)\dot{H}_2\dot{K}+\\
&(36r^7-1152r^3M^4+144r^4M^3+432r^5M^2-234r^6M+864r^2M^5)H_2^{'}K^{''}+\\
&(-36r^6-144r^3M^3-432r^4M^2+1152r^2M^4-864rM^5+234r^5M)H_2^{'}H_2^{'}+\\
&(18r^6M^2-27r^7M+6r^8-72r^4M^4+60r^5M^3)K^{'}\ddot{H}_2^{'}+\\
&(18r^7+480r^2M^5-840r^3M^4+396r^4M^3-93r^6M+66r^5M^2)\dot{H}_2\dot{H}_2^{'}+\\
&(444r^3M^3+42r^4M^2-195r^5M+54r^6-360r^2M^4)H_2\ddot{H}_2+\\
&(-660r^2M^4+364r^3M^3-97r^4M^2+94r^5M-33r^6+192rM^5)\dot{H}_2\dot{H}_2+\\
&\left.(150r^4M^2+120r^3M^3-60r^5M-360r^2M^4)K\ddot{H}_2\right]/r^2(2r+3M)^2(r-2M)\ . \qquad (43)
\end{aligned}
$$

The quantities K and H_2 occurring in (43) are the following combinations of first order metric perturbations:

$$
K \equiv K^{(1)} + (r-2M)\left(\partial_r G^{(1)} - \frac{2}{r^2}h_1^{(1)}\right) \tag{44}
$$

$$
H_2 \equiv H_2^{(1)} + (2r-3M)\left(\partial_r G^{(1)} - \frac{2}{r^2}h_1^{(1)}\right) + r(r-2M)\partial_r\left(\partial_r G^{(1)} - \frac{2}{r^2}h_1^{(1)}\right)\ . \tag{45}
$$

Note that K and H_2 reduce to the metric perturbations $K^{(1)}$ and $H_2^{(1)}$ in the RW gauge.

As explained in connection with (10), it is easy to construct an alternative wave function by adding to $\Psi^{(2)}_{\rm RW}$ terms quadratic in $\Psi^{(1)}$. Any such modification, however, will not have the property that quadratic terms vanish in the Regge-Wheeler gauge, and might make the expression for the source more complicated.

IV. INITIAL DATA

A. Unit normal to the perturbed hypersurfaces

Our spacetime will be foliated, outside the horizon, by spacelike surfaces of constant coordinate time t that, to zero order in ϵ, agree with the surfaces of constant Schwarzschild coordinate time. Our initial hypersurface will be assumed to be one of these constant t surfaces. Viewed as a foliation of a given spacetime, the constant t surfaces will change under coordinate transformation, but our variables $\Psi^{(1)}$ and $\Psi^{(2)}_{\rm RW}$, specified at a particular coordinate value, are invariant under such transformation. More subtle is the fact that for a fixed foliation of a fixed spacetime, the meaning of the partial derivative $\partial/\partial t$ will change when gauge transformations are made that perturbatively change the spatial coordinate labels on a hypersurface. But our quantities are invariant with respect to such perturbative diffeomorphisms on the hypersurfaces, so the $\partial/\partial t$ operation on our quantities will be unchanged.

To relate the time derivatives of our quantities to the extrinsic curvature we must first find, to first and second order, the components of the future directed unit normal to the constant t hypersurfaces. The computation starts with the relationship of the metric and the shift components N_i in the standard ADM [28] decomposition

$$
g_{tr} = N_r(r,t) \tag{46}
$$

$$
g_{t\theta} = N_\theta(r,t) \tag{47}
$$

$$
g_{t\phi} = N_\phi(r,t)\ . \tag{48}
$$

With the definition of the perturbed metric tensor (12)–(18) and the definitions for the shift (46)–(48), the perturbative expressions for the shift components given to second order are

$$
N_r(r,t) = \left[\epsilon H_1^{(1)} + (\epsilon^2/2)H_1^{(2)}\right]P_2(\theta) \tag{49}
$$

$$
N_\theta(r,t) = \left[\epsilon h_0^{(1)} + (\epsilon^2/2)h_0^{(2)}\right]P_2^{'}(\theta) \tag{50}
$$

$$
N_\phi(r,t) = 0\ . \tag{51}
$$

The vanishing of the shift component N_ϕ is due to the assumed axisymmetric character of the collision.

In order to obtain the expression for the perturbed lapse to second order we can use the ADM [28] equality for ${}^{(4)}g_{tt}$,

$$ {}^{(4)}g_{tt} = -N^2 + {}^{(3)}g^{ij} N_i N_j \ . \tag{52} $$

Here summation on i and j ranges over r, θ and ϕ, and ${}^{(3)}g^{ij}$ is the inverse of the 3-metric on a constant t hypersurface. Using (49)–(51) in (52) gives the perturbed lapse to second order

$$ N(r,t) = \sqrt{1-2M/r}\ \left[1 - \frac{1}{2}\epsilon H_0^{(1)} - \frac{1}{2}\epsilon^2 (H_0^{(2)} P_2(\theta) - H_1^{(1)}\, H_1^{(1)} P_2(\theta) P_2(\theta) - \right. $$

$$ \left. \frac{h_0^{(1)} h_0^{(1)} P_2^{'}(\theta) P_2^{'}(\theta)}{r(r-2M)} + \frac{1}{4} H_0^{(1)} H_0^{(1)} P_2(\theta) P_2(\theta) \right] \ . \tag{53} $$

In a similar manner one can compute N^r and N^θ to second order from $N^i = {}^{(3)}g^{ij} N_j$. From these, and N, the covariant n_α and contravariant n^α components of the orthonormal vector to the time slices are given as:

$$ n_\alpha = (-N, 0, 0, 0) \tag{54} $$

$$ n^\alpha = (\frac{1}{N}, -\frac{N^r}{N}, -\frac{N^\theta}{N}, 0) \ . \tag{55} $$

B. Multipole decomposition of extrinsic curvature

In computing the extrinsic curvature K_{ab} on our constant t hypersurfaces, it is useful to start with a tensor harmonic decomposition, similar to that for the metric tensor given in (12)-(18). We define the quantities $K_{rr}^{(i)}$, $K_{r\theta}^{(i)}$, $K_K^{(i)}$, and $K_G^{(i)}$ by the relations

$$ K_{rr} = (1-2M/r)^{-1} \left[\epsilon K_{rr}^{(1)}(r) + \frac{\epsilon^2}{2} K_{rr}^{(2)}(r)\right] P_2(\theta) \tag{56} $$

$$ K_{r\theta} = \left[\epsilon K_{r\theta}^{(1)}(r) + \frac{\epsilon^2}{2} K_{r\theta}^{(2)}(r)\right] P_2^{'}(\theta) \tag{57} $$

$$ K_{\theta\theta} = r^2 \left[\left\{\epsilon K_K^{(1)}(r) + \frac{\epsilon^2}{2} K_K^{(2)}(r)\right\} P_2(\theta) + \left\{\epsilon K_G^{(1)}(r) + \frac{\epsilon^2}{2} K_G^{(2)}(r)\right\} P_2^{''}(\theta)\right] \tag{58} $$

$$ K_{\phi\phi} = r^2 \left[\left\{\epsilon K_K^{(1)}(r) + \frac{\epsilon^2}{2} K_K^{(2)}(r)\right\} \sin^2\theta P_2^{'}(\theta) + \left\{\epsilon K_G^{(1)}(r) + \frac{\epsilon^2}{2} K_G^{(2)}(r)\right\} \sin\theta\cos\theta P_2^{'}(\theta)\right] \ . \tag{59} $$

C. Time derivatives of the metric perturbations

To relate the extrinsic curvature and time derivative of metric perturbations we will need the projector

$$ P_i^\mu = \delta_i^\mu + n_i n^\mu \ . \tag{60} $$

into the 3-dimensional hypersurfaces. The extrinsic curvature is defined, in terms of the unit normal n^α, as

$$ K_{ij} = -\frac{1}{2} P_i^\mu P_j^\nu n_{(\mu;\nu)} \ . \tag{61} $$

Here the symbol "; " means covariant derivative with respect to the 4-metric tensor. With the relations in (46)-(55) the right hand side of (61) can be decomposed into multipoles with coefficients expressed in terms of metric perturbations.

With (61) these coefficients are then related to the multipole coefficients defined in (56) – (59). As examples of the results of this procedure, the time derivatives of $H_2^{(1)}$ and $H_2^{(2)}$ are

$$\partial_t H_2^{(1)} = 2(1-2M/r)\,\partial_r H_1^{(1)} + (2M/r^2)\,H_1^{(1)} + 2\sqrt{1-2M/r}\,K_{rr}^{(1)}\ , \tag{62}$$

$$\begin{aligned}\partial_t H_2^{(2)} = {} & 2(1-2M/r)\,\partial_r H_1^{(2)} + (2M/r^2)\,H_1^{(2)} + 2\sqrt{1-2M/r}\,K_{rr}^{(2)} - (2/7r^4)\ \Big[12M h_1^{(1)} h_0^{(1)} - 6r^2\,h_0^{(1)} H_2^{(1)} + \\ & 12r^2 h_0^{(1)}\partial_r h_1^{(1)} - 24r\,h_0^{(1)}\partial_r h_1^{(1)} - 2r^4\,H_0^{(1)}\partial_r H_1^{(1)} + 2r^4\,H_1^{(1)}\partial_r H_2^{(1)} - 4r^3 M\,H_1^{(1)}\partial_r H_2^{(1)} + \\ & 4r^3 M\,H_0^{(1)}\partial_r H_1^{(1)} + r^4 H_0^{(1)}\partial_t H_2^{(1)} - 2r^2 M\,H_1^{(1)} H_0^{(1)}\Big]\ .\end{aligned} \tag{63}$$

These examples show the general pattern. Time derivatives of first order perturbations of the metric tensor are expressed as linear combinatins of first order metric perturbations. The expressions for time derivatives of second order perturbations contain terms linear in second order metric perturbations, and terms quadratic in first order perturbations.

D. Extrinsic curvature and momentum constraints

Since axisymmetry is assumed, the nontrivial first order momentum constraint equations are $R_{tr}^{(1)} = 0$ and $R_{t\theta}^{(1)} = 0$. The partial derivatives, with respect to time, of the first order metric perturbations can be reexpressed in terms of components of extrinsic curvature by using relations like (62). The resulting, simplified, momentum constraints are

$$0 = r^2(r-2M)\,K_{rr}^{(1)} - r^2(r-2M)\,K_G^{(1)} + r^2(r-2M)\,K_K^{(1)} + (-2r^2+7rM-6M^2)\,K_{r\theta}^{(1)} - r^2(r-2M)^2\,\partial_r K_{r\theta}^{(1)} \tag{64}$$

$$\begin{aligned}0 = {} & -r^2(r-2M)^2\,\partial_r K_K^{(1)} + 3r^2(r-2M)^2\,\partial_r K_G^{(1)} + r(r-2M)^2\,K_{rr}^{(1)} - r(r-2M)^2\,K_K^{(1)} + 3r(r-2M)^2\,K_G^{(1)} \\ & -3(r-2M)^2\,K_{r\theta}^{(1)}\ .\end{aligned} \tag{65}$$

These results will be needed below to help connect $\partial_t \Psi_{\rm RW}^{(2)}$ to the initial value data.

E. Wave function on the initial hypersurface

Cauchy data for the first and second order wave equations requires the values of $\Psi^{(1)}$ and $\Psi_{\rm RW}^{(2)}$ on the initial hypersurface. If the 3-geometry of the initial hypersurface is known, $\Psi^{(1)}$ follows immediately from the definition in (19), and $L^{(2)}$ similarly follows immediately from (24). But the specification of $\Psi_{\rm RW}^{(2)}$ requires $Q_{\rm RW}^{(1)}$, and the definition $Q_{\rm RW}^{(1)}$ in (41) appears to require a knowledge of terms that do not follow directly from the hypersurface 3-geometry. All such terms can be grouped into only two combinations of perturbations:

$$[h_0^{(1)} - \frac{r^2}{2}\partial_t G^{(1)}] \qquad \text{and} \qquad [H_1^{(1)} - \frac{r^2}{2(r-2M)}\partial_t K^{(1)}]\ . \tag{66}$$

These two first order expressions, however, appear in the following components of the extrinsic curvature:

$$K_{\theta\theta}^{(1)} = \frac{1}{\sqrt{1-\frac{2M}{r}}}[\ (h_0^{(1)} - \frac{r^2}{2}\partial_t G^{(1)})P_2^{''}(\theta) + (r-2M)\ (H_1^{(1)} - \frac{r^2}{2(r-2M)}\partial_t K^{(1)})P_2(\theta)] \tag{67}$$

$$K_{\phi\phi}^{(1)} = \frac{1}{\sqrt{1-\frac{2M}{r}}}[\sin\theta\cos\theta\ (h_0^{(1)} - \frac{r^2}{2}\partial_t G^{(1)})P_2^{'}(\theta) + (r-2M)\sin^2\theta\ (H_1^{(1)} - \frac{r^2}{2(r-2M)}\partial_t K^{(1)})P_2(\theta)]\ . \tag{68}$$

An initial value solution in Einstein's theory consists of both the 3-geometry and extrinsic curvature of the initial hypersurface. From an initial value solution, then, the terms in (66) can be evaluated, and the process of specifying $\Psi^{(1)}$ and $\Psi_{\rm RW}^{(2)}$ can be completed.

F. Time derivative of the wave function on the initial hypersurface

The complete specification of Cauchy data for the first and second order Zerilli equations includes the time derivatives $\partial_t\Psi^{(1)}$ and $\partial_t\Psi^{(2)}_{\rm RW}$. These require the time derivatives of metric perturbations, which are found starting with (61), as shown in examples (62) and (63). With this approach the computation of $\partial_t\Psi^{(1)}$ is straightforward. The computation of $\partial_t\Psi^{(2)}_{\rm RW}$, however, requires $\partial_t Q^{(1)}_{\rm RW}$ and is not straightforward. The evaluation of this time derivative produces terms that involve $H_0^{(1)}$ multiplied by time derivatives of groups of extrinsic curvature terms. These groups of terms turn out to be those that occur in the momentum constraint (64), so that the troublesome term is guaranteed to vanish. The resulting, simplified expressions for $\partial_t\Psi^{(1)}$ and $\partial_t\Psi^{(2)}_{\rm RW}$, in terms of hypersurface information, are

$$\begin{aligned}\partial_t\Psi^{(1)} = {} & \frac{2}{21r^4(2r+3M)\sqrt{1-2M/r}}\Big[(28r^6M-28r^5M^2-7r^7)\partial_r K_K^{(1)}+(-84r^6M+21r^7+84r^5M^2)\partial_r K_G^{(1)}+\\ & (28r^4M^2-28r^5M+7r^6)K_{rr}^{(1)}+(168r^4M-168r^3M^2-42r^5)K_{r\theta}^{(1)}+\\ & (14r^6-28r^4M^2-14r^5M)K_K^{(1)}+(21r^5M-42r^4M^2)K_G^{(1)}\Big]\end{aligned} \tag{69}$$

$$\begin{aligned}\partial_t\Psi^{(2)}_{\rm RW} = {} & \frac{2}{21r^4(2r+3M)\sqrt{1-2M/r}}\Big[(28r^6M-28r^5M^2-7r^7)\partial_r K_K^{(2)}+(-84r^6M+21r^7+84r^5M^2)\partial_r K_G^{(2)}+\\ & (28r^4M^2-28r^5M+7r^6)K_{rr}^{(2)}+(168r^4M-168r^3M^2-42r^5)K_{r\theta}^{(2)}+\\ & (14r^6-28r^4M^2-14r^5M)K_K^{(2)}+(21r^5M-42r^4M^2)K_G^{(2)}+\\ & (4r^8-32r^7M-128r^5M^3+96r^6M^2+64r^4M^4)\partial_{r^2}h_1^{(1)}\partial_r K_G^{(1)}+\\ & (4r^7-32r^4M^3-24r^6M+48r^5M^2)\partial_{r^2}K^{(1)}K_{r\theta}^{(1)}+\\ & (-4r^9-80r^5M^4-108r^7M^2+34r^8M+152r^6M^3)\partial_{r^3}G^{(1)}K_G^{(1)}+\\ & (-2r^9+12r^8M+16r^6M^3-24r^7M^2)\partial_{r^2}G^{(1)}\partial_r K_K^{(1)}+\\ & (8r^7+216r^5M^2-68r^6M+160r^3M^4-304r^4M^3)\partial_{r^2}h_1^{(1)}K_G^{(1)}+\\ & (14r^8+22r^7M+464r^5M^3-294r^6M^2-152r^4M^4)\partial_{r^2}G^{(1)}K_G^{(1)}+\\ & (-4r^8-96r^6M^2+32r^7M-64r^4M^4+128r^5M^3)\partial_{r^2}G^{(1)}\partial_r K_{r\theta}^{(1)}+\\ & (-2r^9-24r^7M^2+12r^8M+16r^6M^3)\partial_r K^{(1)}\partial_{r^2}K_G^{(1)}+\\ & (2r^{10}-64r^7M^3+32r^6M^4-16r^9M+48r^8M^2)\partial_{r^2}G^{(1)}\partial_{r^2}K_G^{(1)}+\\ & (-2r^8-4r^7M-48r^5M^3+40r^6M^2)\partial_{r^2}K^{(1)}K_G^{(1)}+\\ & (-4r^8+32r^7M-64r^4M^4+128r^5M^3-96r^6M^2)\partial_r h_1^{(1)}\partial_{r^2}K_G^{(1)}+\\ & (-2r^9-24r^7M^2+12r^8M+16r^6M^3)\partial_r G^{(1)}\partial_{r^2}K_K^{(1)}+\\ & (2r^9-16r^6M^3-12r^8M+24r^7M^2)\partial_{r^3}G^{(1)}K_K^{(1)}+\\ & (-2r^{10}+64r^7M^3+16r^9M-48r^8M^2-32r^6M^4)\partial_{r^3}G^{(1)}\partial_r K_G^{(1)}+\\ & (-112r^6M^3+120r^7M^2-52r^8M+32r^5M^4+8r^9)\partial_r G^{(1)}\partial_{r^2}K_G^{(1)}+\\ & (6r^7M-8r^5M^3-2r^8)\partial_{r^2}G^{(1)}K_K^{(1)}+\\ & (-24r^7M-32r^5M^3+48r^6M^2+4r^8)\partial_{r^2}G^{(1)}K_{rr}^{(1)}+\\ & (24r^7M+32r^5M^3-48r^6M^2-4r^8)K^{(1)}\partial_{r^2}K_G^{(1)}+\\ & (-24r^7M+48r^6M^2-32r^5M^3+4r^8)H_2^{(1)}\partial_{r^2}K_G^{(1)}+\\ & (48r^5M^2-32r^4M^3-24r^6M+4r^7)h_1^{(1)}\partial_{r^2}K_K^{(1)}+\\ & (-48r^5M^2-8r^7-32r^4M^3+40r^6M+64r^3M^4)h_1^{(1)}\partial_{r^2}K_G^{(1)}+\\ & (-80r^4M^3+96r^3M^4+36r^6M-8r^7-24r^5M^2)\partial_{r^2}G^{(1)}K_{r\theta}^{(1)}+\\ & (24r^6M-48r^5M^2+32r^4M^3-4r^7)\partial_{r^2}h_1^{(1)}K_K^{(1)}+\end{aligned}$$

$$
\begin{aligned}
&(96r^6M^3 - 48r^7M^2 + 8r^8M - 64r^5M^4)\partial_{r^2}G^{(1)}\partial_r K_G^{(1)} + \\
&(56r^5M^3 + 30r^7M - 4r^8 - 72r^6M^2)\partial_r G^{(1)}\partial_r K_K^{(1)} + \\
&(24r^6M - 6r^7 - 24r^5M^2)G^{(1)}\partial_r K_K^{(1)} + \\
&(-144r^4M^3 - 14r^7 + 104r^5M^2 + 12r^6M)\partial_r K^{(1)}K_G^{(1)} + \\
&(-144r^5M - 32r^3M^3 + 120r^4M^2 + 96r^2M^4 + 40r^6)h_1^{(1)}\partial_r K_G^{(1)} + \\
&(-16r^5M + 64r^4M^2 - 64r^3M^3)\partial_r K^{(1)}K_{r\theta}^{(1)} + \\
&(-24r^4M^2 + 24r^5M - 6r^6)G^{(1)}K_{rr}^{(1)} + \\
&(24r^4M^3 - 28r^5M^2 - 4r^7 + 16r^6M)\partial_r G^{(1)}K_K^{(1)} + \\
&(504r^4M^3 + 58r^7 - 64r^6M - 356r^5M^2)\partial_r G^{(1)}K_G^{(1)} + \\
&(96r^4M^2 - 48r^5M - 64r^3M^3 + 8r^6)K^{(1)}\partial_r K_{r\theta}^{(1)} + (300r^4M^2 + 42r^5M - 96r^6)G^{(1)}K_G^{(1)} + \\
&(12r^5 + 96r^3M^2 - 48r^2M^3 - 60r^4M)h_1^{(1)}K_{rr}^{(1)} + \\
&(16r^5M^3 - 24r^6M^2 + 12r^7M - 2r^8)\partial_r H_2^{(1)}\partial_r K_G^{(1)} + \\
&(136r^4M + 112r^2M^3 - 24r^5 - 232r^3M^2)h_1^{(1)}K_K^{(1)} + \\
&(-96r^4M^2 - 8r^6 + 64r^3M^3 + 48r^5M)H_2^{(1)}\partial_r K_{r\theta}^{(1)} + \\
&(192r^4M^2 - 64r^5M - 256r^3M^3 + 128r^2M^4 + 8r^6)\partial_r h_1^{(1)}\partial_r K_{r\theta}^{(1)} + \\
&(8r^5M^3 + 8r^6M^2 + 4r^8 - 14r^7M)\partial_r K^{(1)}\partial_r K_G^{(1)} + \\
&(-2r^7 + 8r^5M^2 - 16r^4M^3 + 4r^6M)H_2^{(1)}\partial_r K_G^{(1)} + \\
&(-16r^5M^3 - 12r^7M + 24r^6M^2 + 2r^8)\partial_r G^{(1)}\partial_r K_{rr}^{(1)} + \\
&(16r^5M + 256r^3M^3 + 64r^2M^4 - 240r^4M^2 + 16r^6)\partial_r G^{(1)}K_{r\theta}^{(1)} + \\
&(-144r^4M + 36r^5 + 144r^3M^2)G^{(1)}K_{r\theta}^{(1)} + \\
&(16r^4M^3 + 20r^6M - 4r^7 - 32r^5M^2)K^{(1)}\partial_r K_G^{(1)} + \\
&(-72r^4M + 16r^5 + 48r^3M^2 + 160r^2M^3 - 192rM^4)\partial_r h_1^{(1)}K_{r\theta}^{(1)} + \\
&(-976r^2M^3 + 928r^3M^2 + 12r^5 + 352rM^4 - 288r^4M)h_1^{(1)}K_G^{(1)} + \\
&(64r^2M^3 - 80r^4M + 16r^5 + 96r^3M^2 - 128rM^4)h_1^{(1)}\partial_r K_{r\theta}^{(1)} + \\
&(64r^2M^3 - 24r^5 - 160r^3M^2 + 112r^4M)K^{(1)}K_{r\theta}^{(1)} + (12r^5 + 112r^3M^2 - 64r^2M^3 - 64r^4M)H_2^{(1)}K_{r\theta}^{(1)} + \\
&(-88r^4 + 256r^3M + 96r^2M^2 - 640rM^3 + 256M^4)h_1^{(1)}K_{r\theta}^{(1)} + \\
&(44r^6 + 18r^5M - 212r^4M^2)K^{(1)}K_G^{(1)} + (-6r^7 + 24r^6M - 24r^5M^2)G^{(1)}\partial_r K_G^{(1)} + \\
&(36r^6 - 24r^5M - 96r^4M^2)G^{(1)}K_K^{(1)} + \\
&(-32r^4M^3 + 4r^7 + 48r^5M^2 - 24r^6M)\partial_r h_1^{(1)}\partial_r K_K^{(1)} + \\
&(-64r^3M^4 - 240r^5M^2 + 224r^4M^3 - 16r^7 + 104r^6M)\partial_r G^{(1)}\partial_r K_{r\theta}^{(1)} + \\
&(288r^3M^3 - 276r^4M^2 - 336r^2M^4 + 228r^5M - 60r^6)\partial_r h_1^{(1)}K_G^{(1)} + \\
&(-32r^4M^3 + 48r^5M^2 + 4r^7 - 24r^6M)\partial_r K^{(1)}\partial_r K_{r\theta}^{(1)} + \\
&(88r^4M^3 - 96r^5M^2 + 30r^6M - 2r^7)\partial_r H_2^{(1)}K_G^{(1)} + \\
&(-48r^4M^2 + 24r^5M - 4r^6 + 32r^3M^3)\partial_r H_2^{(1)}K_{r\theta}^{(1)} + \\
&(-40r^4M^3 + 48r^5M^2 - 18r^6M + 2r^7)\partial_r G^{(1)}K_{rr}^{(1)} + \\
&(192r^4M^2 + 20r^6 - 108r^5M - 112r^3M^3)\partial_r h_1^{(1)}K_K^{(1)} + \\
&(48r^4M^2 - 12r^5M - 48r^3M^3)h_1^{(1)}\partial_r K_K^{(1)} + \\
&(28r^6 - 12r^5M - 88r^4M^2)H_2^{(1)}K_G^{(1)} +
\end{aligned}
$$

$$(48r^5M + 64r^3M^3 - 96r^4M^2 - 8r^6)\partial_r h_1^{(1)} K_{rr}^{(1)} +$$
$$(-176r^5M^3 + 132r^6M^2 + 16r^4M^4 - 8r^7M - 8r^8)\partial_r G^{(1)} \partial_r K_G^{(1)} +$$
$$(-16r^7 + 112r^6M - 288r^5M^2 + 320r^4M^3 - 128r^3M^4)\partial_r h_1^{(1)} \partial_r K_G^{(1)} +$$
$$(32r^3M^3 + 24r^5M - 48r^4M^2 - 4r^6) h_1^{(1)} \partial_r K_{rr}^{(1)} + (-6r^6 + 12r^5M) H_2^{(1)} K_K^{(1)} +$$
$$(-8r^6M + 2r^7 + 8r^5M^2)\partial_r H_2^{(1)} K_K^{(1)}\Big] \ . \tag{70}$$

V. ENERGY RADIATED

Radiation is most clearly analyzed in a reference system that is asymptotically flat (AF). By this we shall mean a coordinate system in which the deviations of the metric $\delta g_{\mu\nu}$ from Minkowski form decrease with r, at constant $t - r^*$, according to:

$$\delta g_{tt}, \delta g_{tr}, \delta g_{rr} \sim \mathcal{O}(r^{-3}) \qquad \delta g_{tr}, \delta g_{r\theta} \sim \mathcal{O}(r^{-1})$$

$$\delta g_{\theta\theta} + \delta g_{\theta\phi}/\sin^2\theta \sim \mathcal{O}(r^1) \qquad \delta g_{\theta\phi}, \delta g_{\theta\theta} - \delta g_{\theta\phi}/\sin^2\theta \sim \mathcal{O}(r^3) \ . \tag{71}$$

In such a coordinate system, the power carried by gravitational waves is given by [29,30]

$$\frac{d\text{Power}}{d\Omega} = \frac{1}{16\pi r^2}\left[\frac{1}{\sin^2\theta}\left(\frac{\partial g_{\theta\phi}}{\partial t}\right)^2 + \frac{1}{4}\left(\frac{\partial g_{\theta\theta}}{\partial t} - \frac{1}{\sin^2\theta}\frac{\partial g_{\phi\phi}}{\partial t}\right)^2\right] \tag{72}$$

In the statement of the conditions for an AF gauge, and in (72), there is no reference to the order of the metric perturbations in some expansion parameter ϵ. To compute the power to first order in ϵ, one finds the perturbations $\delta g_{\theta\theta}, \delta g_{\theta\phi}, \delta g_{\phi\phi}$ in a gauge that satisfies the AF conditions to first order in ϵ and uses those values in (72). For a computation correct to second order, the values of $\delta g_{\theta\theta}, \delta g_{\theta\phi}, \delta g_{\phi\phi}$ used in (72) must in principle be computed in a gauge that satisfies the AF conditions to second order in ϵ.

In practice, the use of gauge invariant quantities $\Psi^{(1)}$ and $\Psi^{(2)}_{\rm RW}$ in our computations simplifies the evaluation of energy. A first and second order gauge transformation can always be done to bring the metric perturbations into a form that is AF to first and second order, and $\Psi^{(1)}$ and $\Psi^{(2)}_{\rm RW}$ are unaffected by such a transformation. We can, therefore, treat them as if they had been computed in a gauge that is first and second order AF, and we need only read off the first and second order metric perturbations from $\Psi^{(1)}$ and $\Psi^{(2)}_{\rm RW}$. When restricted to first order only, this is the method that was used to compute radiated power in Refs. [6] and [30].

If we take the expressions in (17) and (18) to be in the AF gauge, then for our axisymmetric quadrupole example the expression in (72) becomes

$$\frac{d\text{Power}}{d\Omega} = \frac{r^2}{64\pi}\left(\epsilon\frac{\partial G^{(1)}}{\partial t} + \frac{\epsilon^2}{2}\frac{\partial G^{(2)}}{\partial t}\right)^2\left(P_2^{''} - \cot\theta P_2^{'}\right)^2 = \frac{9r^2}{64\pi}\left(\epsilon\frac{\partial G^{(1)}}{\partial t} + \frac{\epsilon^2}{2}\frac{\partial G^{(2)}}{\partial t}\right)^2 \sin^4\theta \ . \tag{73}$$

The first order part is easily dealt with, and the treatment is identical to that in equations (III-21) – (III-26) of Ref. [30]. From the definition of $\Psi^{(1)}$ in (19) and the conditions in (71) it follows that in the AF gauge

$$12rG^{(1)} = \Psi^{(1)} + \mathcal{O}(r^{-1}) \ . \tag{74}$$

Evaluating the second order part is considerably more difficult. Using the AF conditions in (71), one must solve (24) for $G^{(2)}$. The result will contain terms that are linear in second order perturbations and quadratic in first order perturbations. The linear terms are identical to those in the first order analysis, so the result can be written in the form

$$12rG^{(2)} = \Psi^{(2)}_{\rm RW} - Q^{(1)}_{\rm RW} + \mathcal{O}(r^{-1}) \ . \tag{75}$$

Here $Q^{(1)}_{\rm RW}$ is the set of terms quadratic in first order perturbations, as defined in (8), and as explicitly exhibited in (41). It remains to find $Q^{(1)}_{\rm RW}$ to $\mathcal{O}(r^0)$ in the AF gauge. The result must be expressible entirely in terms of $\Psi^{(1)}$,

since (73) is an expression for a physical quantity, and $\Psi^{(1)}$ contains all the gauge invariant first order information about perturbations.

To find $Q^{(1)}_{\rm RW}$ in the AF gauge, we start by writing the following asymptotic expansions for first order perturbative quantities:

$$\begin{aligned}
H_0^{(1)} &= f_0/r^3 + \cdots \\
H_1^{(1)} &= f_1/r^3 + \cdots \\
H_2^{(1)} &= f_2/r^3 + \cdots \\
h_0^{(1)} &= f_3/r + I_0/r^2 + \cdots \\
h_1^{(1)} &= f_4/r + I_1/r^2 + \cdots \\
K^{(1)} &= f_5/r + f_6/r^2 + f_7/r^3 + \cdots \\
G^{(1)} &= f_8/r + f_9/r^2 + f_{10}/r^3 + \cdots ,
\end{aligned} \tag{76}$$

where $f_1 \cdots f_{10}, I_0, I_1$ are functions of $t - r^*$. When (76) is used in the expression for $Q^{(1)}_{\rm RW}$ given in (41), the result is an expression of the form

$$Q^{(1)}_{\rm RW} = \mathcal{Q}_0 + r\mathcal{Q}_1 + r^2\mathcal{Q}_2 + \mathcal{O}(r^{-1}) \ , \tag{77}$$

where $\mathcal{Q}_0, \mathcal{Q}_1, \mathcal{Q}_2$ are functions of $t - r^*$. As an example, we present here the explicit expression for $\mathcal{Q}_2$:

$$\mathcal{Q}_2 = -2f_8'f_5'' + 10f_8'f_8'' \ , \tag{78}$$

where a prime ($'$) denotes differentiation with respect to $t - r_*$. The expressions for $\mathcal{Q}_0$ and $\mathcal{Q}_1$ are very lengthy and will not be explicitly exhibited.

The radiated power is a physical quantity. Since $\Psi^{(1)}$ and $\Psi^{(2)}_{\rm RW}$ carry all the (first and second order) gauge invariant information about the spacetime, it must be possible to express the coefficients $\mathcal{Q}_0, \mathcal{Q}_1, \mathcal{Q}_2$ in terms of $\Psi^{(1)}$. To find such expressions we start by using the AF expansions of (76) in Einstein's vacuum equations to derive

$$3f_8' - f_5' = 0 \tag{79}$$

$$6Mf_4' - 3f_{10}' + 7Mf_5 - 2f_6 - 3I_1' + f_7' - 9Mf_8 - 3f_3 - 3I_0' = 0 \tag{80}$$

$$f_5 - f_8 - f_9' = 0 \tag{81}$$

$$f_3' + f_4' + f_9' = 0 \tag{82}$$

$$4f_{10}' - 4Mf_4' + 2I_1' + 2f_9 - 2Mf_8 - 4Mf_9' + 2f_4 + 2I_0' = 0 \tag{83}$$

$$f_6'' - 3f_9'' = 0 \tag{84}$$

$$2Mf_5' - f_6' + 2f_7'' - 9Mf_8' \ . \tag{85}$$

The first two equations are respectively the r^6 and r^4 parts of the Einstein equation $G_{tt} + G_{tr} = 0$; Eqs. (81), (82), (83) are combinations of the r^4 and r^5 parts of $G_{\theta\theta} = 0$ and $G_{\phi\phi} = 0$; Eq. (84) is the r^5 part of $G_{rr} = 0$; Eq. (85) is a combination of the r^3 parts of $G_{\theta\theta} = 0$ and $G_{\theta\theta} = 0$, and the r^5 part of $G_{rr} = 0$. We use (82) in the form $f_3 + f_4 + f_9 = 0$. The justification for this is that f_3, f_4 and f_9 must be zero in a stationary solution [31]. The integration constant must be zero therefore, when (82) is integrated. Similar arguments justify integration of (84). With (79) –(85), and considerable manipulation, we end up with

$$\mathcal{Q}_0 = \frac{1}{6048}\left(36\Psi^{(1)}\partial_t\Psi^{(1)} - 19M\partial_t\Psi^{(1)}\partial_t\Psi^{(1)}\right) \tag{86}$$

$$\mathcal{Q}_1 = \frac{1}{1008}\left(7\,\partial_t\Psi^{(1)}\partial_t\Psi^{(1)} - 4M\,\Psi^{(1)}\partial_{tt}\Psi^{(1)}\right) \tag{87}$$

$$\mathcal{Q}_2 = \frac{1}{126}\,\partial_t\Psi^{(1)}\partial_{tt}\Psi^{(1)}\ . \tag{88}$$

Notice that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ do not vanish, and hence $Q^{(1)}_{\rm RW}$ diverges at $r\to\infty$. It is important to understand that this is not incompatible with with asymptotic flatness. The value of $G^{(2)}$ must fall off as r^{-1} in AF coordinates, so $Q^{(1)}_{\rm RW}$ can diverge if there is a compensating divergence in $\Psi^{(2)}_{\rm RW}$. There must, in fact, be a divergence of this order, since the source term in (42) turns out to diverge. In practice, numerical computations with divergent quantities are to be avoided. It is useful, therefore, to exploit the fact that the second order wave function is not unique. (See the discussion at the end of Sec. I.) We now introduce an alternative second order wave function $\Psi^{(2)}_{\rm rad}$ by

$$\Psi^{(2)}_{\rm rad} = \Psi^{(2)}_{\rm RW} + \Xi_{\rm rad}\ , \tag{89}$$

where

$$\Xi_{\rm rad} \equiv -\frac{1}{2016}\Big(144\Psi^{(1)}\partial_t\Psi^{(1)} - 76M\partial_t\Psi^{(1)}\partial_t\Psi^{(1)}$$

$$+r\left[56\,\partial_t\Psi^{(1)}\partial_t\Psi^{(1)} - 32M\,\Psi^{(1)}\partial_{tt}\Psi^{(1)}\right] + 16\,r^2\partial_t\Psi^{(1)}\partial_{tt}\Psi^{(1)}\Big)\ . \tag{90}$$

The wave equation for $\Psi^{(2)}_{\rm rad}$,

$$\widehat{Z}(\Psi^{(2)}_{\rm rad}) = \mathcal{S}_{\rm RW} + \widehat{Z}(\Xi_{\rm rad}) \equiv \mathcal{S}_{\rm rad}\ , \tag{91}$$

has a source term that is well behaved at $r\to\infty$, and in AF coordinates, we then have from (75) and (86)– (90), that

$$12rG^{(2)} = \Psi^{(2)}_{\rm rad} + \mathcal{O}(r^{-1})\ , \tag{92}$$

and thus

$$\frac{d\text{Power}}{d\Omega} = \frac{1}{1024\pi}\left(\epsilon\frac{\partial\Psi^{(1)}}{\partial t} + \frac{\epsilon^2}{2}\frac{\partial\Psi^{(2)}_{\rm rad}}{\partial t}\right)^2\sin^4\theta\ . \tag{93}$$

Integration over all angles then gives us the total power

$$\text{Power} = \frac{1}{480}\left(\epsilon\frac{\partial\Psi^{(1)}}{\partial t} + \frac{\epsilon^2}{2}\frac{\partial\Psi^{(2)}_{\rm rad}}{\partial t}\right)^2\ . \tag{94}$$

VI. SUMMARY AND DISCUSSION

For convenience, we repeat and summarize here the main results of the paper. The wavefunction

$$\Psi^{(1)} = \frac{r}{6\,(2r+3M)}\left[2(r-2M)(H_2^{(1)} - r\partial_r K^{(1)}) - 2(r-3M)K^{(1)} + 6\left\{rK^{(1)} + \frac{(r-2M)}{r}(r^2\partial_r G^{(1)} - 2h_1^{(1)})\right\}\right]\ , \tag{95}$$

is Moncrief's [23] wavefunction. It is constructed completely from first order perturbations of the 3-geometry on a hypersurface, and its time derivative can be found from the first order perturbations of the 3-geometry and extrinsic curvature of the hypersurface. This wave function is invariant with respect to first order gauge transformations, and satisfies the Zerilli equation

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_*^2} + V(r)\right]\Psi^{(1)} = 0\ , \tag{96}$$

where $V(r)$ is given in (23). From (24), (89),(90), our second order wave function is

$$\Psi^{(2)}_{\rm rad} = \frac{r}{6\,(2r+3M)}\Big[2(r-2M)(H_2^{(2)} - r\partial_r K^{(2)}) - 2(r-3M)K^{(2)} + \tag{97}$$

$$6\left\{rK^{(2)} + \frac{(r-2M)}{r}(r^2\partial_r G^{(2)} - 2h_1^{(2)})\right\}\Big] - \frac{1}{2016}\Big(144\Psi^{(1)}\partial_t\Psi^{(1)} - 76M\partial_t\Psi^{(1)}\partial_t\Psi^{(1)}$$

$$+r\left[56\,\partial_t\Psi^{(1)}\partial_t\Psi^{(1)} - 32M\,\Psi^{(1)}\partial_{tt}\Psi^{(1)}\right] + 16\,r^2\partial_t\Psi^{(1)}\partial_{tt}\Psi^{(1)}\Big) + Q^{(1)}_{\rm RW}(\Psi^{(1)},\Psi^{(1)})\ , \tag{98}$$

where the explicit form of $Q^{(1)}_{\rm RW}$ is given in (41). The wave function $\Psi^{(2)}_{\rm rad}$ gives the second order equivalent of the two important advantages of the Moncrief function $\Psi^{(1)}$: (i) it is gauge invariant, as spelled out in Sec. III and (ii) its value and its time derivative can be found directly from the first and second order perturbations of the 3-geometry and extrinsic curvature of a hypersurface, as described in Sec. IV. The second order wave function satisfies a Zerilli equation

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_*^2} + V(r)\right]\Psi^{(2)}_{\rm rad} = \mathcal{S}_{\rm rad}\ . \tag{99}$$

The source term $\mathcal{S}_{\rm rad}$ is well behaved at infinity, and is given by (42), (90) and (91). The outgoing solution for $\Psi^{(2)}_{\rm rad}$ has the asymptotic behavior of a function of retarded time $t - r^*$, and the gravitational wave power carried in the perturbations is given by the simple prescription in (93).

The scheme outlined in the above paragraph gives a formalism that is definitive and complete (for the case of even parity, axisymmetric, quadrupole perturbations). This formalism involves only gauge invariant variables, and requires the solution of a wave equation with a source that is well behaved at spatial infinity. Due to the gauge invariant nature of the wave function, the Cauchy data for the wave equation can be constructed immediately from an initial value solution given in any gauge.

The method given here constitues a reformulation of the fixed-gauge approach used by Gleiser *et al.* [14–19]. The gauge invariant approach appears to offer advantages both in the organization of computations, and in the conceptual clarity of the gauge invariant variables.

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